Principled Domain Extension for Analytic IK

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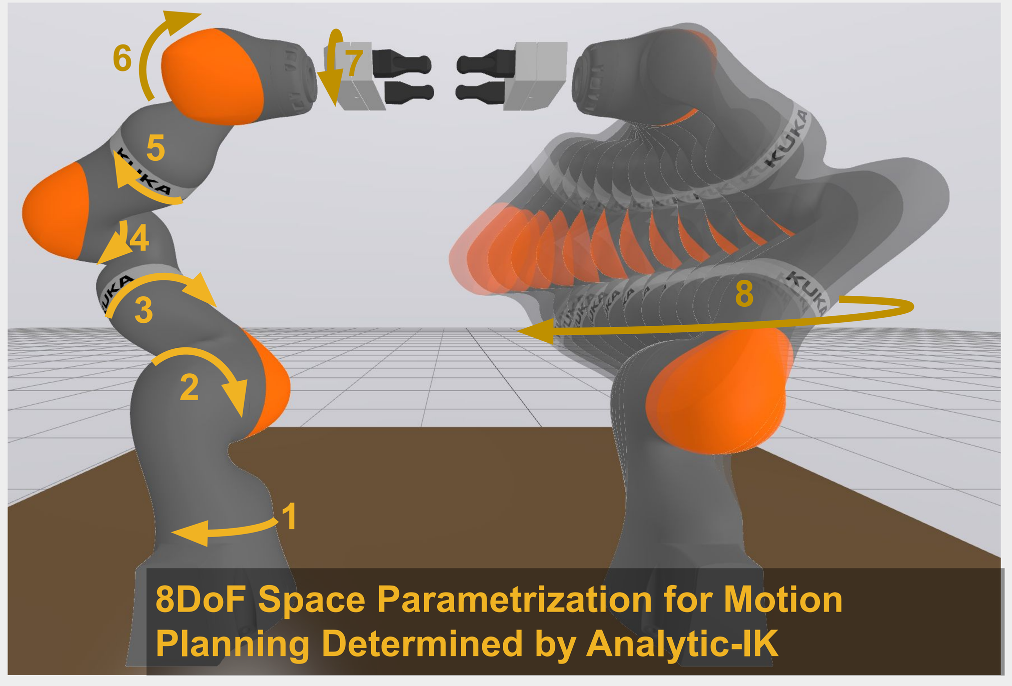
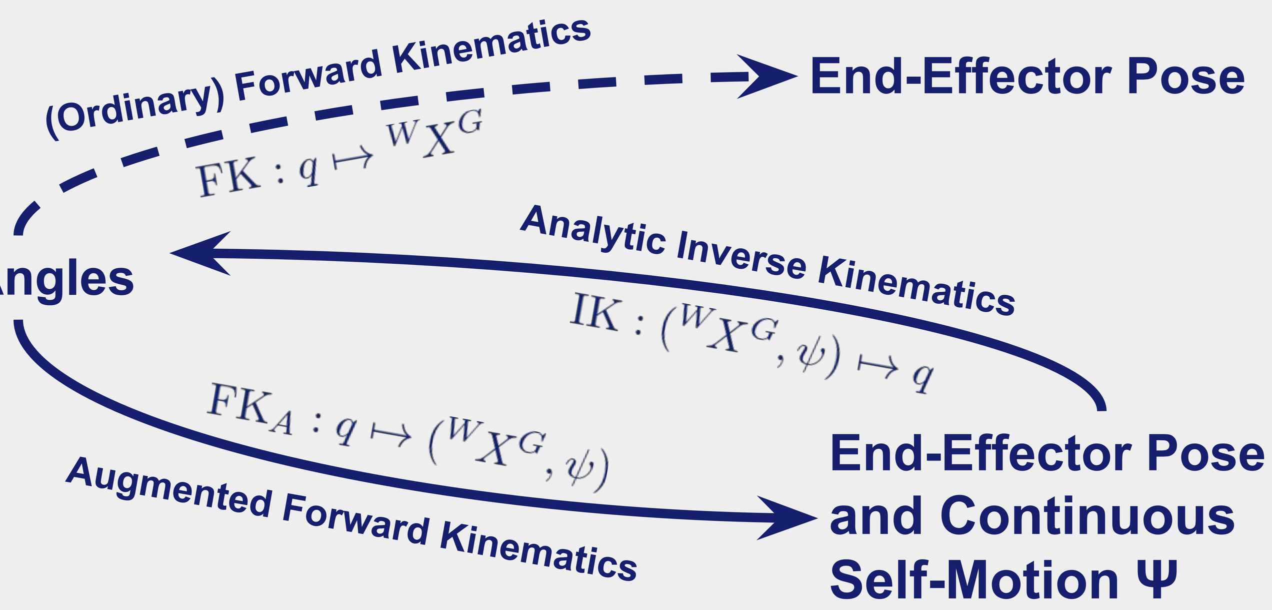
1. Analytic IK Parametrizations

Analytic IK has been used for motion planning and trajopt under end-effector constraints, partially due to its differentiability [3].



8DoF Space Parametrization for Motion Planning Determined by Analytic-IK

- ***Self-motion resolves the "many-to-one" issue of IK.***
- ***Augmented FK is the inverse of the analytic IK function.***



- **Existing solutions must be written to support automatic differentiation, *limiting us to manipulators with simple hand-written analytic IK solutions.***
- **We want a method to differentiate through *any* analytic IK solution, including those computed by black-box solvers (e.g. IKFast [1]).**

We can use the inverse function theorem to compute gradients of black-box analytic IK functions!

Procedure:

1. $q := \mathrm{IK}({}^W X^G, \psi)$
2. $J_A := \nabla \mathrm{FK}_A(q)$
3. $\nabla \mathrm{IK}({}^W X^G, \psi) = J_A^{-1}$

Can be computed efficiently by solving a linear system!

But this approach has problems!

- **Gradients (and values!) are undefined for non-reachable EE configs!**
- **Optimizers do not guarantee feasibility of intermediate iterates!**

2. Extending the Domain of IK

Key Idea: use gradient from the closest reachable configuration!

$$\hat{\mathrm{IK}}({}^W X^G, \psi) = \begin{cases} \arg\min_q \|\mathrm{FK}_A(q) - ({}^W X^G, \psi)\|^2 & {}^W X^G \text{ nonreachable} \\ \mathrm{IK}({}^W X^G, \psi) & {}^W X^G \text{ reachable} \end{cases}$$

Differentiate at nonreachable configurations via sensitivity analysis

$J_A(q) := \nabla \mathrm{FK}_A(q)$ is the augmented kinematic Jacobian

r_i is the ith component of the residual $r := ({}^W X^G, \psi) - \mathrm{FK}_A(q)$

$H_i(q)$ is the ith component of the augmented kinematic Hessian

$$\nabla \hat{\mathrm{IK}}({}^W X^G, \psi) = \left(J_A(q)^T J_A(q) + \sum_{i=1}^{n} r_i H_i(q) \right)^{-1} J_A(q)^T$$

- **Can leverage a variety of numerical approximations**
- **Alternatively, calculate kinematic Hessian analytically**

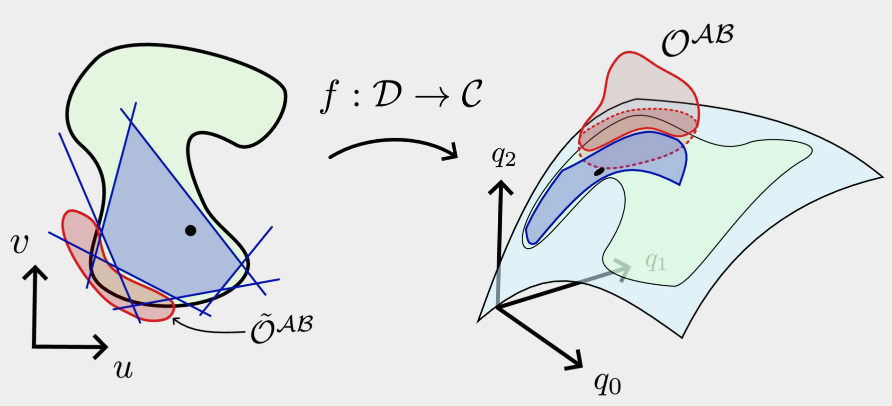
3. Reachability Constraints

Reachability constraints enforce IK correctness for optimizers.

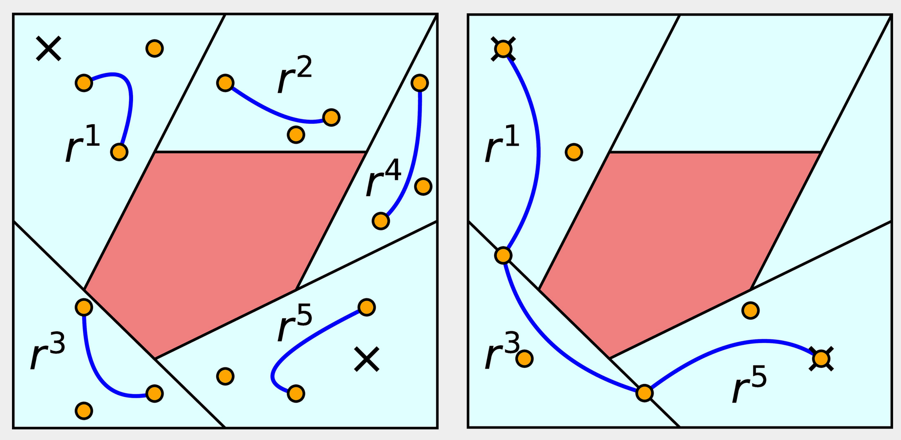
1. **Direct reachability [4]:** $\|\mathrm{FK}(\mathrm{IK}({}^W X^G, \psi)) - {}^W X^G\|_F^2 \leq 0$
   👍 **Requires no special knowledge of IK solution internals**
   👎 **Active on relative interior of the feasible set**
2. **Probing functions [5]:** $\arccos(\alpha) \Rightarrow -1 \leq \alpha \leq 1$
   👎 **Requires knowledge and modification of IK solution internals**
   👍 **Only active on relative boundary of the feasible set**
3. **(NEW) Boundary reachability:** $-\log\det(JJ^T + \epsilon I) \leq \tau$
   Geometric fact: boundary of the reachable workspace is singular!
   👍 **Requires no special knowledge of IK solution internals**
   👍 **Only active on relative boundary of the feasible set**
   (J is the ordinary kinematic Jacobian, not the augmented Jacobian.)

4. Downstream Experiments

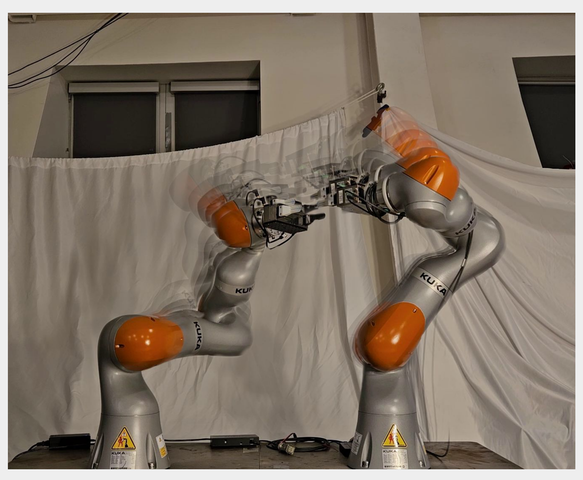
We evaluate runtime performance of analytic IK + IFT gradients in a variety of downstream applications:



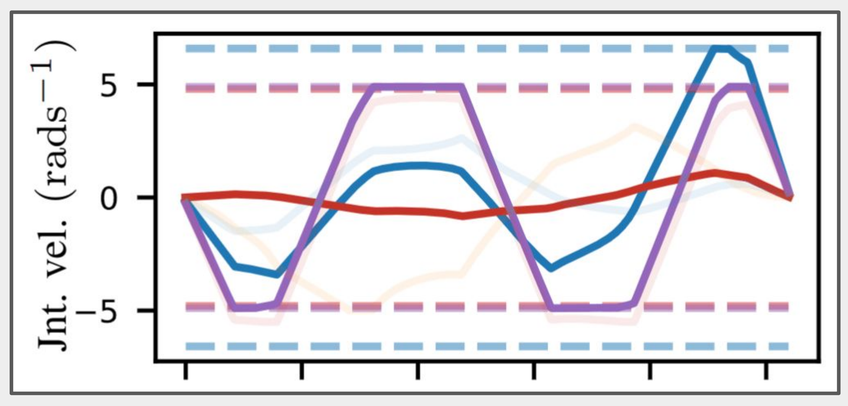
Region generation with IRIS-NP2 [2]



motion planning with GCS [3]



Kinematic Trajectory Optimization [4]



Time-Optimal Path Parameterization [6]

Gradient Method	IrisNp2	Trajopt	TOPPRA
Baselines (AD methods)			
Autodiff, Old Reach	1.47	1.15	2.45
Autodiff, New Reach	1.40	0.99	2.52
Autodiff, Boundary Reach	1.31	2.56	2.37
IFT with Direct Reach			
IFT, zero gradients	5.28	1.69	3.12
IFT, pseudoinverse	5.08	1.58	3.14
IFT, LM Constant	3.70	0.88	3.06
IFT, LM SVT	3.39	1.39	3.04
IFT, Newton	3.33	1.16	3.15
IFT, Resid. Std	2.21	1.05	3.33
IFT, Resid. Aniso	2.20	1.64	3.42
IFT with Boundary Reach			
IFT, LM Constant (Boundary)	2.48	3.00	3.12
IFT, zero gradients (Boundary)	1.81	3.24	3.15
IFT, Resid. Aniso (Boundary)	1.74	3.85	3.48
IFT, pseudoinverse (Boundary)	1.67	3.07	3.19
IFT, Newton (Boundary)	1.65	3.13	3.48
IFT, Resid. Std (Boundary)	1.63	2.91	3.53
IFT, LM SVT (Boundary)	1.59	2.74	3.13

Runtime (seconds)

With appropriate numerical regularization of gradients, IFT performance approaches the hand-designed autodiff solution.

References

[1] R. Diankov, 2010. Automated construction of robotic manipulation programs. PhD Dissertation.
[2] Cohn, T., Werner, P. and Tedrake, R., 2025. Faster Algorithms for Growing Collision-Free Regions of Constrained Bimanual Configuration Spaces. In: IROS 2025 Workshop on Frontiers in Dynamic, Intelligent, and Adaptive Multi-Arm Manipulation.
[3] Marcucci, T., Petersen, M., Von Wrangel, D. and Tedrake, R., 2023. Motion planning around obstacles with convex optimization. Science robotics, 8(84), p.eadf7843.
[4] Cohn, T., Shaw, S., Simchowitz, M. and Tedrake, R., 2024, May. Constrained bimanual planning with analytic inverse kinematics. In 2024 IEEE International Conference on Robotics and Automation (ICRA) (pp. 6935-6942). IEEE.
[5] Cohn, T., Tang, L., Amice, A. and Tedrake, R., 2026. A Framework for Combining Optimization-Based and Analytic Inverse Kinematics. arXiv preprint arXiv:2602.05092.
[6] Pham, H. and Pham, Q.C., 2018. A new approach to time-optimal path parameterization based on reachability analysis. IEEE Transactions on Robotics, 34(3), pp.645-659.

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