Planning along Differentiable Charts of Constraint Manifolds with the Inverse Function Theorem

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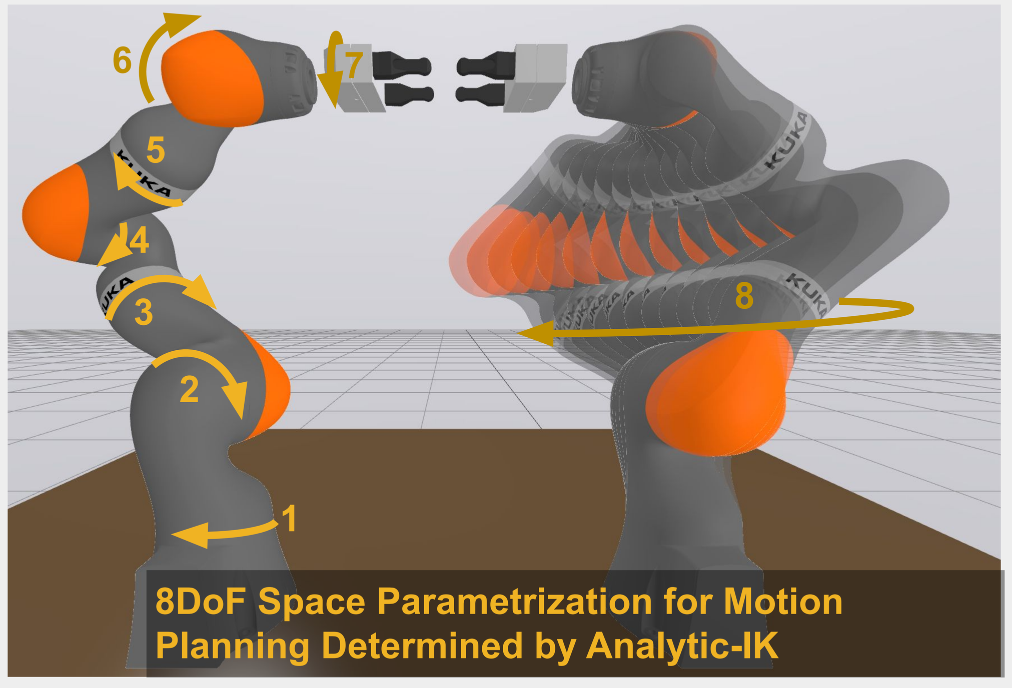
1. Analytic IK Parametrizations

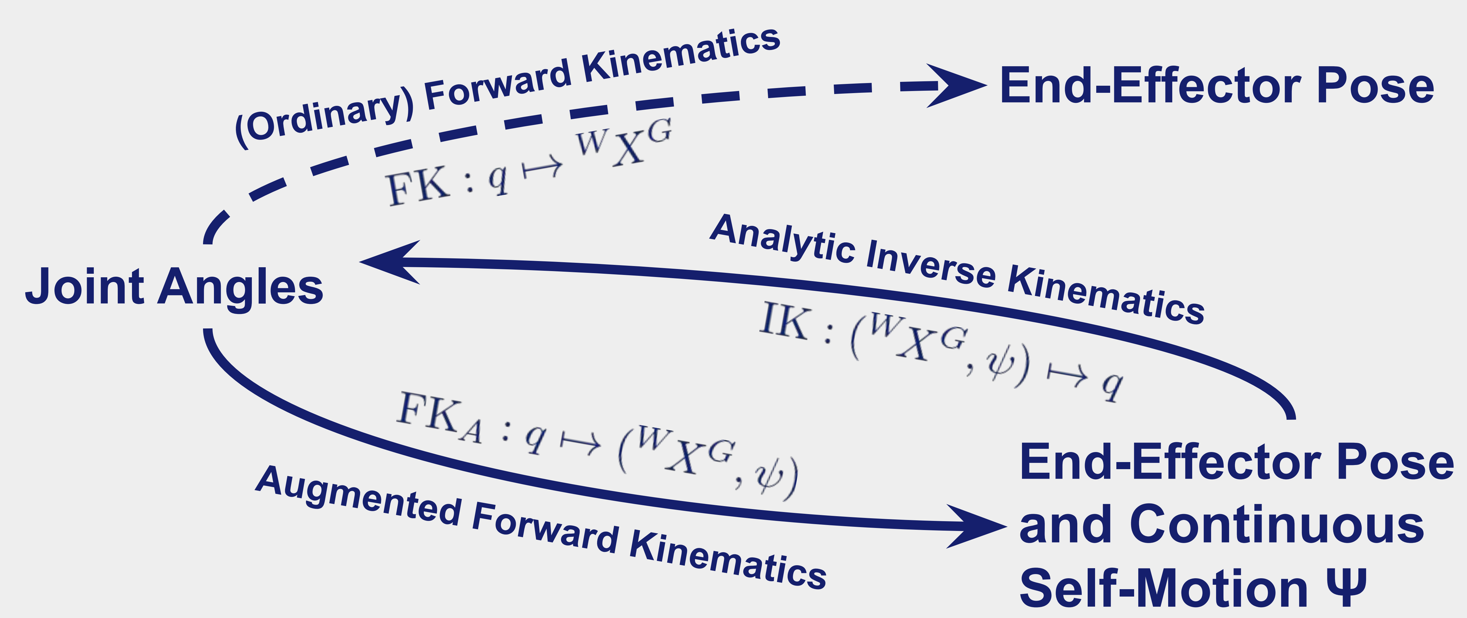
Analytic IK has been used for motion planning and trajopt under end-effector constraints, partially due to their differentiability [3].



- **Existing solutions must be written to support automatic differentiation, limiting us to manipulators with simple hand-written analytic IK solutions.**
- **We want a method to differentiate through *any* analytic IK solution, including those computed by black-box solvers (e.g. IKFast [1]).**

2. Augmented Forward Kinematics

- ***Forward kinematics is not an injection; it can't invert directly.***
- ***Self-motion resolves the "many-to-one" issue of IK.***
- ***Augmented FK is the inverse of the analytic IK function.***



Key idea: avoid autodiff with the inverse function theorem!

Procedure:

1. $q := \mathrm{IK}({}^W X^G, \psi)$
2. $J := \nabla \mathrm{FK}_A(q)$
3. $\nabla \mathrm{IK}({}^W X^G, \psi) = J^{-1}$

Can be computed efficiently by solving a linear system!

3. Extending the Domain of IK

- **Gradients are not well defined for non-reachable EE configs.**
- **Many optimizations start with infeasible guesses, so we need some gradient defined outside the domain.**

Key Idea: use gradient from the closest reachable configuration!

$$\hat{\mathrm{IK}}({}^W X^G, \psi) = \begin{cases} \arg\min_q \| \mathrm{FK}_A(q) - ({}^W X^G, \psi) \|^2 & {}^W X^G \text{ nonreachable} \\ \mathrm{IK}({}^W X^G, \psi) & {}^W X^G \text{ reachable} \end{cases}$$

Differentiate at nonreachable configurations via sensitivity analysis

$J_A(q) := \nabla \mathrm{FK}_A(q)$ is the augmented kinematic Jacobian

r_i is the ith component of the residual $r := ({}^W X^G, \psi) - \mathrm{FK}_A(q)$

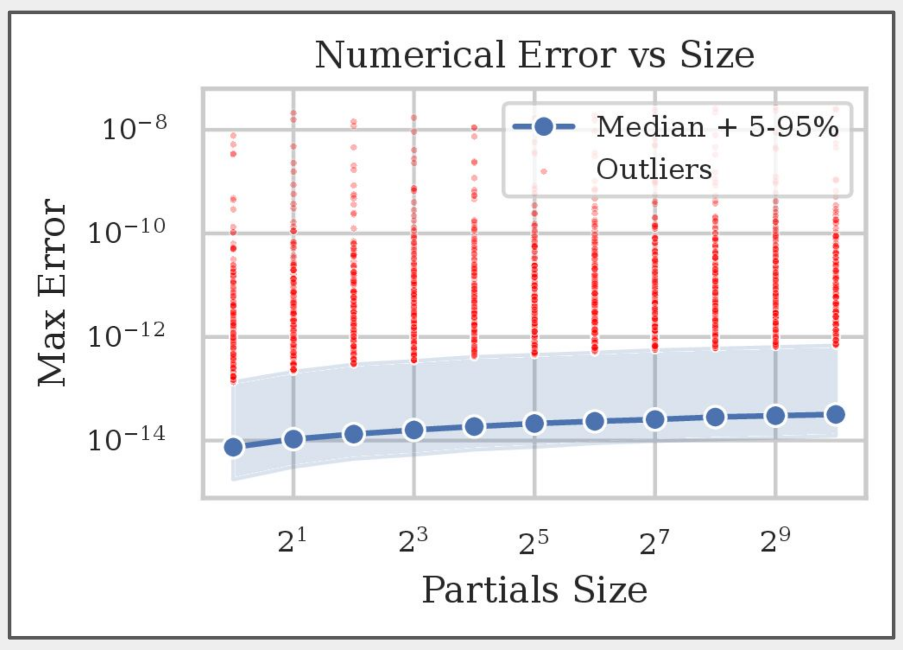
$H_i(q)$ is the ith component of the augmented kinematic Hessian

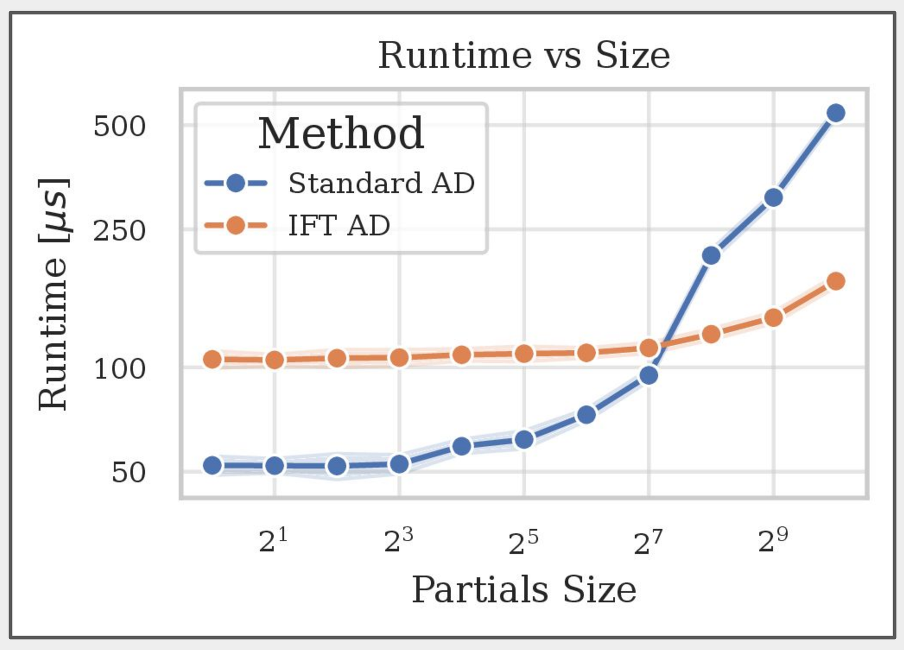
$$\nabla \hat{\mathrm{IK}}({}^W X^G, \psi) = \left(J_A(q)^T J_A(q) + \sum_{i=1}^{n} r_i H_i(q) \right)^{-1} J_A(q)^T$$

- **Can leverage a variety of numerical approximations**
- **Alternatively, calculate kinematic Hessian analytically**

4. Numerical Experiments

Procedure: sample a random reachable configuration and partials vector, compare autodiff with inverse function theorem gradients.



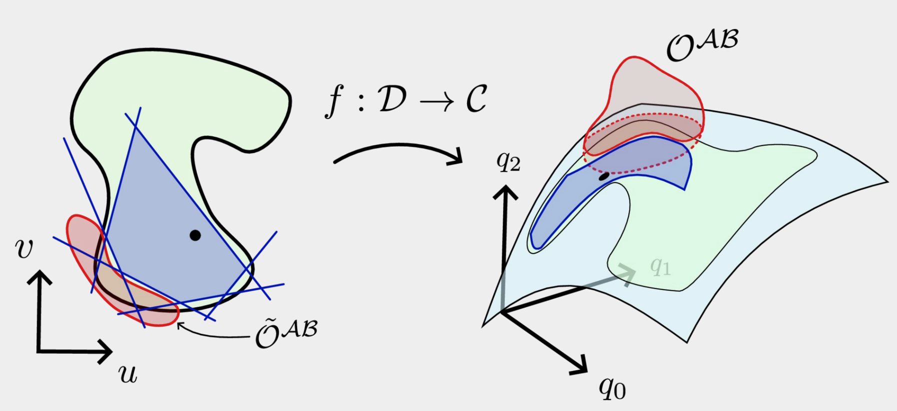


Solutions are numerically accurate.

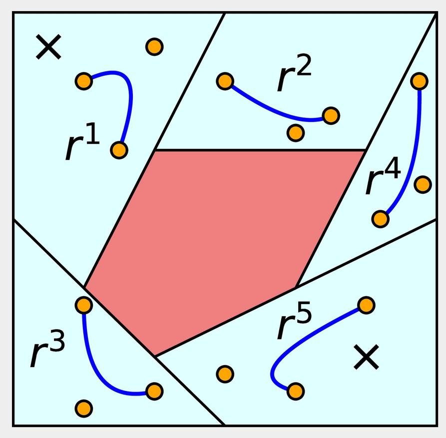
IFT is slower for small partials vector size (constructing the Jacobian is a constant factor), but scale better.

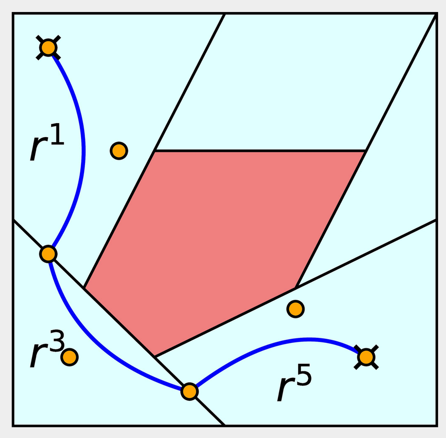
5. Downstream Experiments

We evaluate runtime performance of analytic IK + IFT gradients in a variety of downstream applications:

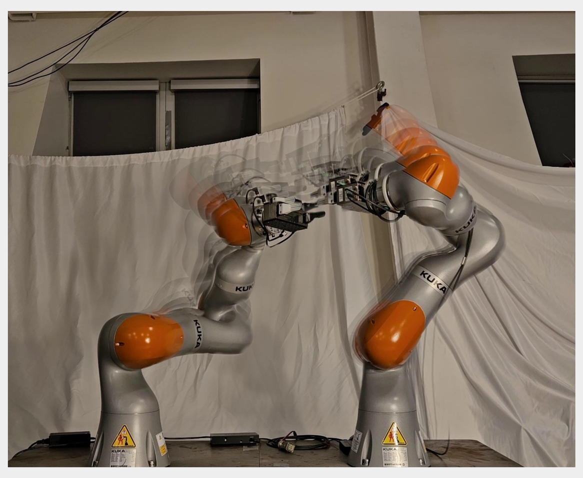


Region generation with IRIS-NP2 [2]

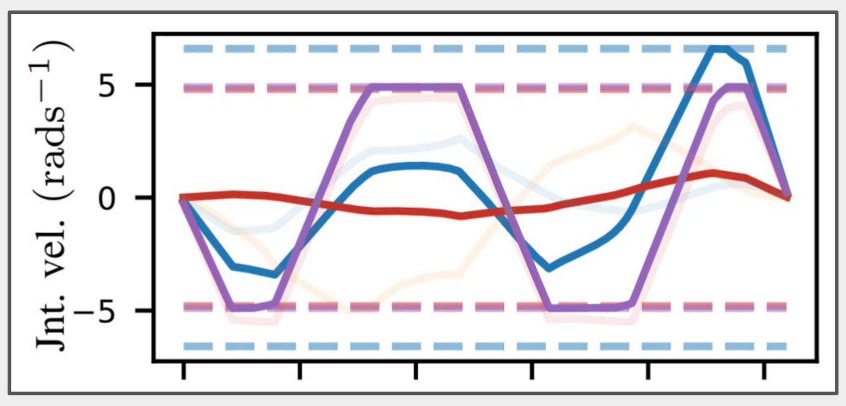




motion planning with GCS [3]



Kinematic Trajectory Optimization [4]



Time-Optimal Path Parameterization [5]

Gradient Method	IrisNp2	Trajopt	TOPPRA
Baselines (AD methods)			
Autodiff, Old Reach	25.02	1.15	2.45
Autodiff, New Reach	23.76	0.99	2.52
Autodiff, Boundary Reach	22.32	2.56	2.37
IFT with Direct Reach			
IFT, zero gradients	89.76	1.69	3.12
IFT, pseudoinverse	86.41	1.58	3.14
IFT, LM Constant	62.94	0.88	3.06
IFT, LM SVT	57.63	1.39	3.04
IFT, Newton	56.64	1.16	3.15
IFT, Resid. Std	37.53	1.05	3.33
IFT, Resid. Aniso	37.34	1.64	3.42
IFT with Boundary Reach			
IFT, LM Constant (Boundary)	42.15	3.00	3.12
IFT, zero gradients (Boundary)	30.69	3.24	3.15
IFT, Resid. Aniso (Boundary)	29.59	3.85	3.48
IFT, pseudoinverse (Boundary)	28.45	3.07	3.19
IFT, Newton (Boundary)	28.11	3.13	3.48
IFT, Resid. Std (Boundary)	27.76	2.91	3.53
IFT, LM SVT (Boundary)	27.04	2.74	3.13

Runtime (seconds)

With appropriate numerical regularization of gradients, IFT performance approaches the hand-designed autodiff solution.

References

[1] R. Diankov, 2010. Automated construction of robotic manipulation programs. PhD Dissertation.
[2] Cohn, T., Werner, P. and Tedrake, R., 2025. Faster Algorithms for Growing Collision-Free Regions of Constrained Bimanual Configuration Spaces. In: IROS 2025 Workshop on Frontiers in Dynamic, Intelligent, and Adaptive Multi-Arm Manipulation.
[3] Marcucci, T., Petersen, M., Von Wrangel, D. and Tedrake, R., 2023. Motion planning around obstacles with convex optimization. Science robotics, 8(84), p.eadf7843.
[4] Cohn, T., Shaw, S., Simchowitz, M. and Tedrake, R., 2024, May. Constrained bimanual planning with analytic inverse kinematics. In 2024 IEEE International Conference on Robotics and Automation (ICRA) (pp. 6935-6942). IEEE.
[5] Pham, H. and Pham, Q.C., 2018. A new approach to time-optimal path parameterization based on reachability analysis. IEEE Transactions on Robotics, 34(3), pp.645-659.